A superlinearly convergent subgradient method for sharp semismooth problems

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Joint work with Damek Davis

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Setting

Goal: fast first-order algorithms for

$$\operatorname*{argmin}_{x \in \mathbb{R}^d} f(x) \quad \text{where} \quad \min f = 0,$$

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Classical example: Hoffman bound for LPs / linear inequalities.[1]

$$\operatorname{dist}(x, \{x \mid Ax \leq b\}) \leq H_A \|(Ax - b)_+\|$$

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$$\text{find } \ x \ \text{ s.t. } \ F(x) = 0 \ \Leftrightarrow \ \operatorname*{argmin}_x \|F(x)\|.$$

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Growth condition known as *metric subregularity*:[2]

$$\|F(x)\| \geq \mu \operatorname{dist}(x, \mathcal{X}_*).$$

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Set intersection problems: given $\mathcal{X}_1, \mathcal{X}_2$ closed,

$$\text{find } \bar{x} \in \mathcal{X}_1 \cap \mathcal{X}_2 \;\Leftrightarrow\; \operatorname*{argmin}_{x} \left\{ \operatorname{dist}(x, \mathcal{X}_1) + \operatorname{dist}(x, \mathcal{X}_2) \right\}.$$

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Growth condition known as *linear regularity*:

$$\operatorname{dist}(x, \mathcal{X}_1) + \operatorname{dist}(x, \mathcal{X}_2) \geq \mu \operatorname{dist}(x, \mathcal{X}_1 \cap \mathcal{X}_2).$$

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Fast algorithms?

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Sharp growth and rapid convergence

Subgradient method:

$$x_{k+1} := x_k - \alpha_k \frac{v_k}{\|v_k\|}, \quad v_k \in \underbrace{\partial f(x_k)}_{\text{Clarke subdifferential}}.$$

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Theorem: $\{x_k\}$ converge *linearly* with *Polyak step size*:

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- Classically known for convex functions[1].
- Recently generalized for *weakly convex* functions[2].

Key example : $f = (\text{convex}) \circ (\text{smooth})$.

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Question: *Faster than linear convergence using only subgradients?*

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An answer in pictures

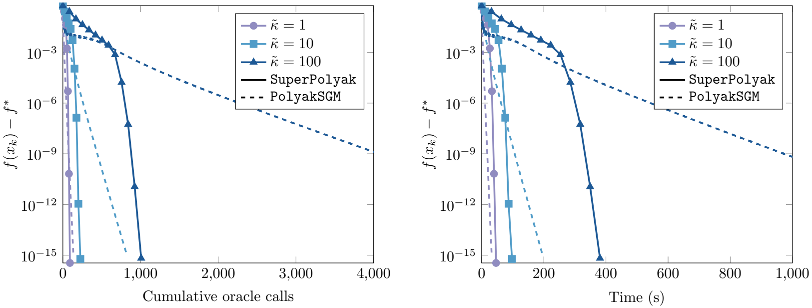

Figure: Our algorithm (`SuperPolyak`) applied to a matrix sensing problem with dimensions $(d, r) = (2^{15}, 2)$ and $m = 2^{19}$ measurements. Here, $\tilde{\kappa}$ is the condition number of the unknown matrix.

Algorithm converges superlinearly, with fewer subgradient oracle calls. **How?**

Motivation: Polyak bundle

Simplifying assumption: problem has *unique solution* $\bar{x}$.

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Polyak step equivalent to:

$$x_{k+1} := \operatorname*{argmin}_{x} \left\{ \|x - x_k\|^2 \mid f(x_k) + \langle v_k, x - x_k \rangle \leq 0 \right\}, \; v_k \in \partial f(x_k).$$

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To improve convergence, can try to use a **bundle** $(y_i, v_i \in \partial f(y_i))$:

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- **Note**: $\bar{x}$ *always feasible* when f is convex.
- Bundle points y_i chosen among the past k iterates.
- Each step requires solving a QP.

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Guarantees?

- Good practical performance, but rate similar to subgradient method.[3]

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Our approach

Main idea.

- Replace QP by a linear system:

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Problem admits **closed-form solution**:

$$x_{k+1} = x_k - A^\dagger \begin{bmatrix} f(y_0) + \langle v_0, x_k - y_0 \rangle \\ \vdots \\ f(y_i) + \langle v_i, x_k - y_i \rangle \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} v_0^\top \\ \vdots \\ v_i^\top \end{bmatrix}.$$

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Key additional assumption: **Semismoothness.**

$$f(x) + \langle v, \bar{x} - x \rangle = o(\|x - \bar{x}\|), \quad \text{for } v \in \partial f(x) \text{ and as } x \to \bar{x}.$$

$\Rightarrow$ Implies $\bar{x}$ is *nearly feasible* for system of equations!

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Semismoothness is common. Satisfied by:

- Convex and smooth, and (convex) $\circ$ (smooth) functions.
- Any *semialgebraic* function.[1]

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Question: How to choose the bundle points $\{y_i\}$?

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Algorithm `PolyakBundle`(x, τ)

$y_0 := x$; $v_0 \in \partial f(y_0)$; $A_1 = [v_0^\mathsf{T}]$.

for $i = 1, \ldots, d$ **do**

$$y_i := y_0 - A_i^\dagger \left[f(y_j) + \langle v_j, y_0 - y_j\rangle\right]_{j=0}^{i-1}; \quad A_{i+1} := \begin{bmatrix} A_i \\ v_i^\mathsf{T} \end{bmatrix} \text{ for } v_i \in \partial f(y_i)$$

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Note: first bundle point recovers Polyak subgradient step:

$$y_1 = y_0 - (v_0^\top)^\dagger (f(y_0) + \langle v_0, y_0 - y_0 \rangle) = y_0 - \frac{f(y_0)}{\|v_0\|^2} v_0.$$

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3. Since $A_j \in \mathbb{R}^{(j+1) \times d}$, superlinear improvement achieved within d steps.

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Theorem: C. & Davis, 2022 (informal)

Assume x near $\bar{x}$ and τ is sufficiently large. Then

$$f(\texttt{PolyakBundle}(x, \tau)) = o(f(x)).$$

- ✓ Provable early termination strategies available.
- ✓ Sequence of linear systems solved incrementally (QR-based algorithm).

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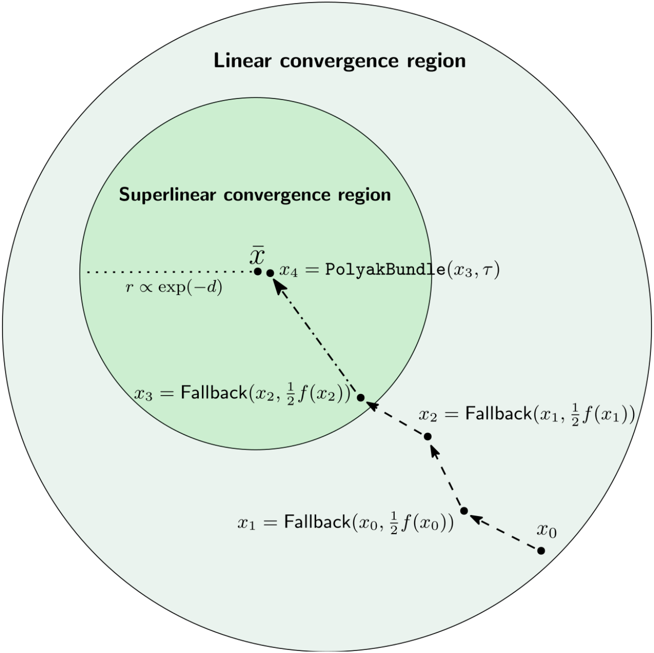
- ✗ **Issue**: region of local convergence $\propto \exp(-d)$.
  - **Fix:** couple with linearly convergent method (e.g., subgradient).

High-level overview



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Question: which algorithm can we use as fallback method?

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Theorem: **C.** and Davis, 2022 (informal)

`SuperPolyak` with the Polyak subgradient method as fallback enters the region of superlinear convergence in $O(d)$ iterations, as long as x_0 is in a dimension-independent region around $\bar{x}$.

Consequences: root-finding problems

Problem: find x s.t. $F(x) = 0$.

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Question: is superlinear convergence possible *without* invertibility condition?

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Assumptions. Suppose that F satisfies:

1. **Semismoothness**: for all x near $\bar{x}$ and $A(x) \in \partial F(x)$,

$$\|F(x) + A(\bar{x} - x)\| = o(\|x - \bar{x}\|)$$

2. **Metric subregularity**: for x near $\bar{x}$,

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Under above assumptions, `SuperPolyak` converges locally superlinearly.

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$$z_{i+1} := T(z_i), \quad \text{where} \quad T := I - F.$$

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Example: LASSO

Goal: recover s-sparse $x_\sharp \in \mathbb{R}^d$ from $y = Ax_\sharp \in \mathbb{R}^m$ $(m \ll d)$.

$$\underset{x \in \mathbb{R}^d}{\operatorname{argmin}} \left\{ \frac{1}{2} \|Ax - b\|^2 + \lambda \|x\|_1 \right\}$$

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Fallback algorithm: proximal gradient (ISTA).

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Note: $I - T$ metrically subregular but need not satisfy invertibility condition!

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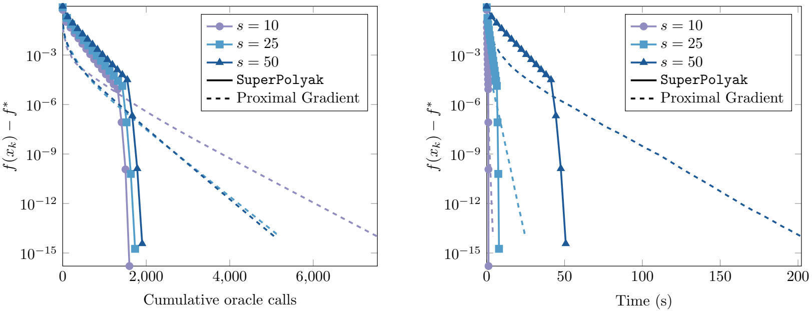

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Problem: find $\bar{x} \in \mathcal{X}_* = \mathcal{X}_1 \cap \mathcal{X}_2$, $\mathcal{X}_1$ and $\mathcal{X}_2$ closed.

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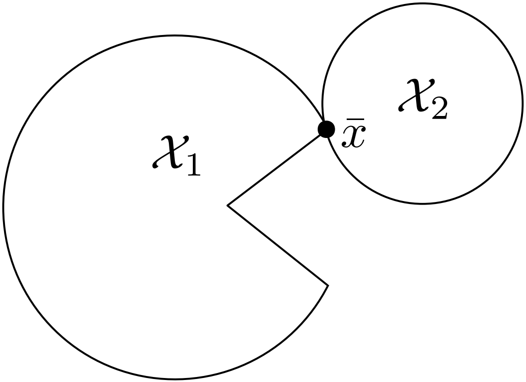

Figure: Semialgebraic sets intersecting at a single point $\bar{x}$.

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Setting I: intersections of semialgebraic sets.

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$$\mathrm{dist}(x, \mathcal{X}_1) + \mathrm{dist}(x, \mathcal{X}_2) \geq \mu \, \mathrm{dist}(x, \mathcal{X}_*) \text{ for all } x \text{ near } \bar{x}.$$

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Related work: QP-based algorithm that converges under similar conditions.[3]

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Example: complex phase retrieval

Complex phase retrieval: given $y_\sharp \in \mathbb{R}^m$ with $(y_\sharp)_i = |\langle a_i, x_\sharp \rangle|$:

$$\text{find} \quad \hat{y} \in \mathcal{Y}_1 \cap \mathcal{Y}_2, \quad \mathcal{Y}_1 := \{u \in \mathbb{C}^m \mid |u| = y\}, \; \mathcal{Y}_2 := \mathrm{Range}(A).$$

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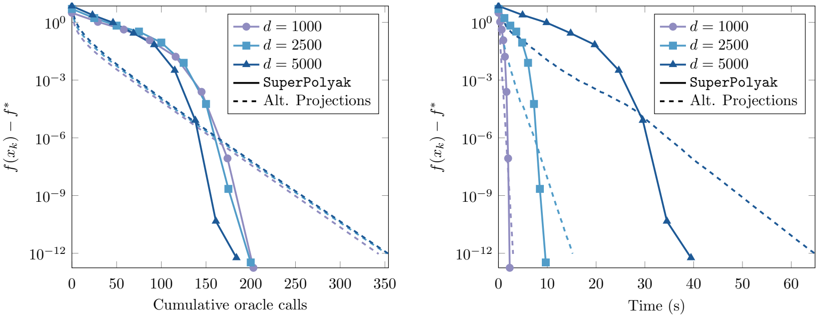

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Concluding remarks

Not covered in this talk:

- Results for non-isolated $\mathcal{X}_*$ via a uniformization of semismoothness.[5]
- Provable early termination strategies for `PolyakBundle` loop.
    - In practice, lead to small (even constant-sized) linear systems.

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Open questions:

1. Reduce dim. dependence of local convergence region of `PolyakBundle`.
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Thank you!

`arXiv:abs/2201.04611`

[5] Davis et al. '21

Example: low-rank matrix sensing

Bilinear sensing: recover low-rank factors $U_\sharp, V_\sharp$ from bilinear measurements:

$$y_i = \ell_i^\mathsf{T} U_\sharp V_\sharp^\mathsf{T} r_i, \quad \ell_i, r_i \overset{\text{iid}}{\sim} \mathcal{N}(0, I_d).$$

[6] **C.** et al., '21

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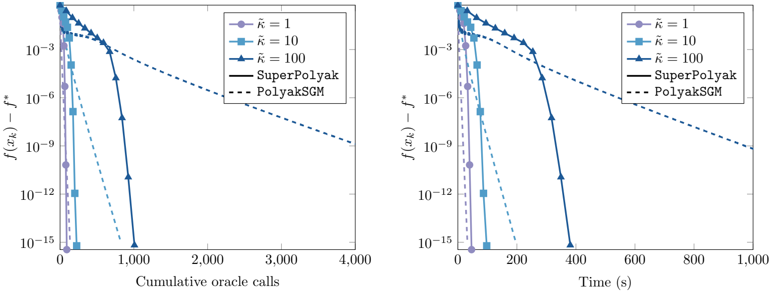

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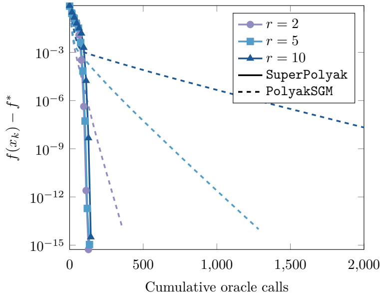

From projection problem to linear system:

$$\begin{aligned}
&\operatorname{argmin}\left\{\|x - x_k\|^2 \mid f(y_i) + \langle v_i, x - y_i\rangle = 0\right\} \\
&= \operatorname{argmin}\left\{\|x - x_k\|^2 \mid \langle v_i, x - x_k\rangle = \langle v_i, y_i - x_k\rangle - f(y_i)\right\} \\
&= \operatorname{argmin}\left\{\|z\|^2 \mid \langle v_i, z\rangle = \langle v_i, y_i - x_k\rangle - f(y_i)\right\} \\
&= \operatorname{argmin}\left\{\|z\|^2 \mid Az + \left[f(y_i) + \langle v_i, x_k - y_i\rangle\right]_i = 0\right\}
\end{aligned}$$

Least-norm solution of $Ax + b = 0$: $x = -A^{\dagger} b$.